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Hankel Spaces of Entire Functions*

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A generalization of the Hankel transformation is obtained from the expansion theorem for Hilbert spaces of vector valued entire functions [1] analogous to the Hilbert spaces of complex valued entire functions appearing in the theory of the Hankel transformation [2].

The coefficient space $\mathscr{C}$ is taken to be the two dimensional Hilbert space of column vectors in the Euclidean norm. A vector is always an element of this space. An operator is a two-by-two matrix with complex entries. A bar is used to denote the conjugate transpose of a rectangular matrix. Let $A(z)$ and $B(z)$ be operator valued entire functions such that $B(z)\,\bar{A}(\bar{z}) = A(z)\,\bar{B}(\bar{z})$, such that $A(z) - iB(z)$ has invertible values in the upper half-plane, and such that $A(z) + iB(z)$ has invertible values in the lower half-plane. Assume that $K(w, w) \geqslant 0$ for all complex w, where

$$K(w, z) = [B(z)\,\bar{A}(w) - A(z)\,\bar{B}(w)]/[\pi(z - \bar{w})].$$

Let $\mathscr{H}(A, B)$ be the set of vector valued entire functions $F(z)$ such that

$$\|F(t)\|^2 = \int_{-\infty}^{+\infty} |[A(t) - iB(t)]^{-1} F(t)|^2\, dt < \infty,$$

such that $[A(z) - iB(z)]^{-1} F(z)$ is of bounded type and of nonpositive mean type in the upper half-plane, and such that $[A(z) + iB(z)]^{-1} F(z)$ is of bounded type and of nonpositive mean type in the lower half-plane. Then $\mathscr{H}(A, B)$ is a Hilbert space which contains $K(w, z)\, c$ as a function of z for every vector c and complex number w. The identity $\bar{c}F(w) = \langle F(t), K(w, t)\, c\rangle$ holds for every element $F(z)$ of the space.

The inclusion theory for spaces $\mathscr{H}(A, B)$ uses Hilbert spaces whose elements are pairs of vector valued entire functions. The notation I is used for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ considered as a matrix of operators. Let

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

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be a matrix whose elements are operator valued entire functions of z. Assume that the identity

$$M(z)\, I\bar{M}(\bar{z}) = I = \bar{M}(\bar{z})\, IM(z)$$

and the inequality

$$[M(z)\, I\bar{M}(z) - I]/(z - \bar{z}) \geqslant 0$$

are satisfied for all complex z. Then there exists a unique Hilbert space $\mathscr{H}(M)$ whose elements are pairs $\binom{F_+(z)}{F_-(z)}$ of vector valued entire functions such that the expression

$$\frac{M(z)\, I\bar{M}(w) - I}{2\pi(z - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to the space for all choices of vectors u and v and for all complex numbers w, and such that the identity

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)\, I\bar{M}(w) - I}{2\pi(t - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

holds for every element $\binom{F_+(z)}{F_-(z)}$ of the space.

If $\mathscr{H}(A(a), B(a))$ and $\mathscr{H}(M(a, b))$ are given spaces, a space $\mathscr{H}(A(b), B(b))$ exists such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\, M(a, b, z).$$

The space $\mathscr{H}(A(a), B(a))$ is contained in the space $\mathscr{H}(A(b), B(b))$ and the inclusion does not increase norms. If there is no nonzero constant $\binom{u}{v}$ in $\mathscr{H}(M(a, b))$ such that $A(a, z)\, u + B(a, z)\, v$ belongs to $\mathscr{H}(A(a), B(a))$, then $\mathscr{H}(A(a), B(a))$ is contained isometrically in $\mathscr{H}(A(b), B(b))$. The transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[A(a, z)\, F_+(z) + B(a, z)\, F_-(z)]$$

is an isometry of $\mathscr{H}(M(a, b))$ onto the orthogonal complement of $\mathscr{H}(A(a), B(a))$ in $\mathscr{H}(A(b), B(b))$.

If $\mathscr{H}(A(a), B(a))$ and $\mathscr{H}(A(b), B(b))$ are given spaces such that $\mathscr{H}(A(a), B(a))$ is contained isometrically in $\mathscr{H}(A(b), B(b))$, and if the transformation $F(z) \to F(w)$ takes $\mathscr{H}(A(a), B(a))$ onto $\mathscr{C}$ for every complex number w, then the identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\, M(a, b, z)$$

holds for a unique matrix $M(a, b, z)$ of operator valued entire functions such that a space $\mathscr{H}(M(a, b))$ exists.

A construction of spaces $\mathcal{H}(M)$ is obtained using any continuous, nondecreasing function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \bar{\beta}(t) & \gamma(t) \end{pmatrix}$$

of $t > 0$ whose values are self-adjoint matrices of operators. For each number $a > 0$ and for each complex number w, there exists a unique continuous function

$$M(a, t, w) = \begin{pmatrix} A(a, t, w) & B(a, t, w) \\ C(a, t, w) & D(a, t, w) \end{pmatrix}$$

of $t \geqslant a$, whose values are matrices of operators, such that the integral equation

$$M(a, b, w)\, I - I = w \int_a^b M(a, t, w)\, dm(t)$$

holds when $b \geqslant a$. A space $\mathcal{H}(M(a, b))$ exists, $M(a, b, 0)$ is the identity matrix, and $M'(a, b, 0)\, I = m(b) - m(a)$. The identity

$$M(a, c, z) = M(a, b, z)\, M(b, c, z)$$

holds when $a < b < c$.

The spaces appearing in the theory of the Hankel transformation admit homogeneous substitutions as isometries.

THEOREM 1. *Let ν be a given operator, $\nu + \bar{\nu} = 0$. Let $\mathcal{H}(A, B)$ be a given space, $A(0) = 1$ and $B(0) = 0$, such that the transformation $F(z) \to F(0)$ takes the space onto $\mathcal{C}$. Assume that the function $a^{\frac{1}{2}+\frac{1}{2}\nu} F(az)$ belongs to the space whenever $F(z)$ belongs to the space, $0 < a < 1$, and that it always has the same norm as $F(z)$. Then the equations*

$$zB'(z) + \tfrac{1}{2}\, \nu B(z) + \tfrac{1}{2}\, B(z)\, \bar{\nu} = zA(z)\, \alpha'(1) + zB(z)\, \beta'(1)$$

and

$$- zA'(z) - \tfrac{1}{2}\, \nu A(z) + \tfrac{1}{2}\, A(z)\, \nu - B(z)\, r'(1) = zA(z)\, \beta'(1) + zB(z)\, \gamma'(1)$$

are satisfied for a nonnegative matrix

$$\begin{pmatrix} \alpha'(1) & \beta'(1) \\ \bar{\beta}'(1) & \gamma'(1) \end{pmatrix}$$

of operators with no nonzero vector in the kernel of $\alpha'(1)$ and for a self-adjoint operator $r'(1)$.

Another fundamental property of the Hankel transformation is given by its relationship to a second-order differential operator.

THEOREM 2. *In Theorem* 1 *assume that* $r'(1) = 0$, *that the functions* $\nu^2 F(z)$ *and* $zF''(z) + (1+\nu)F'(z)$ *belong to the space whenever* $F(z)$ *belongs to the space, and that the identities*

$$\langle \nu^2 F(t), G(t)\rangle = \langle F(t), \nu^2 G(t)\rangle$$

and

$$\langle tF''(t) + (1+\nu)F'(t), G(t)\rangle = \langle F(t), tG''(t) + (1+\nu)\, G'(t)\rangle$$

hold for all elements $F(z)$ *and* $G(z)$ *of the space. Then* ν^2 *commutes with* $A(z)$ *and* $B(z)$, *and there exist operators* U *and* V, *commuting with* ν^2, *such that* $\bar{U}V = \bar{V}U$, *and such that the equations*

$$\alpha'(1) = U\bar{U}, \qquad \beta'(1) = U\bar{V}, \qquad \gamma'(1) = V\bar{V}$$

are satisfied. The function $\Phi(z) = A(z)\, U + B(z)\, V$ *has the power series expansion*

$$U - \frac{1}{1!\,(1+\nu)}\, U\left(\frac{\bar{V}\nu U + \bar{U}\bar{\nu}V}{2}\, z\right) + \frac{1}{2!\,(1+\nu)\,(2+\nu)}\, U\left(\frac{\bar{V}\nu U + \bar{U}\bar{\nu}V}{2}\, z\right)^2 - \cdots .$$

There exists a unique Hankel space corresponding to any desired choice of the parameters U and V.

THEOREM 3. *Let* ν *be a given operator,* $\nu + \bar{\nu} = 0$. *If* U *and* V *are given operators,* U *invertible and* $\bar{U}V = \bar{V}U$, *there exist unique operator valued entire functions* $A(z)$ *and* $B(z)$ *such that* $A(0) = 1$ *and* $B(0) = 0$ *and such that the equations*

$$zB'(z) + \tfrac{1}{2}\,\nu B(z) + \tfrac{1}{2}\, B(z)\,\bar{\nu} = z[A(z)\, U + B(z)\, V]\,\bar{U}$$

and

$$-\, zA'(z) - \tfrac{1}{2}\,\nu A(z) + \tfrac{1}{2}\, A(z)\,\nu = z[A(z)\, U + B(z)\, V]\,\bar{V}$$

are satisfied. A space $\mathscr{H}(A, B)$ *exists, and it satisfies the hypotheses of Theorems* 1 *and* 2.

The Hankel expansion is a convolution over the multiplicative group of the real line.

THEOREM 4. *If, in Theorem 3, $f(t)$ is a square integrable, vector valued function of $t > 0$ which vanishes for $t > 1$, then*

$$\pi F(z) = \int_0^\infty t^{\frac{1}{2}\nu} \Phi(tz) f(t)\, dt$$

belongs to $\mathscr{H}(A, B)$ and

$$\pi \| F(t) \|^2 = \int_0^\infty | f(t)|^2\, dt.$$

Every element of $\mathscr{H}(A, B)$ is of this form. Let $f(t)$ and $g(t)$ be square integrable, vector valued functions of $t > 0$ which vanish for $t > 1$, and let $F(z)$ and $G(z)$ be the corresponding elements of $\mathscr{H}(A, B)$. The condition $G(z) = zF(z)$ is necessary and sufficient that $f(t)$ be (equivalent to) a differentiable function of $t > 0$ with absolutely continuous derivative such that

$$g(t) = -tf''(t) - f'(t) + \tfrac{1}{4}\nu^2 f(t)/t$$

almost everywhere and such that $\int_0^\infty t^{\frac{1}{2}\nu} Ug(t)\, dt = 0$. The condition $G(z) = a^{\frac{1}{2}+\frac{1}{2}\nu} F(az)$ is necessary and sufficient that $g(t) = a^{-\frac{1}{2}} f(t/a)$ almost everywhere. The condition $G(z) = -zF''(z) - (1 + \nu) F'(z)$ is necessary and sufficient that

$$g(t) = \tfrac{1}{2}(\bar{V}\nu U + \bar{U}\bar{\nu} V)\, tf(t)$$

almost everywhere. The condition $G(z) = \nu^2 F(z)$ is necessary and sufficient that $g(t) = \nu^2 f(t)$ almost everywhere.

The construction is of particular interest when the operator $(\bar{V}\nu U + \bar{U}\bar{\nu} V)/2$ is unitary. The notation $F(c; z)$ is used for the doubly confluent hypergeometric series

$$1 + \frac{z}{1!\, c} + \frac{z^2}{2!\, c(c+1)} + \cdots .$$

THEOREM 5. *If, in Theorem 3,*

$$\nu = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}$$

for a nonzero number λ, if

$$U = \begin{pmatrix} 1 & \omega_+ \\ 1 & \omega_- \end{pmatrix}$$

for distinct numbers ω_+ *and* ω_- *of absolute value one, and if*

$$i\lambda(\bar{\omega}_+ - \bar{\omega}_-)\, V = \begin{pmatrix} \bar{\omega}_- & 1 \\ \bar{\omega}_+ & 1 \end{pmatrix},$$

then

$$\Phi(z) = \begin{pmatrix} F(1+i\lambda; -z) & \omega_+ F(1+i\lambda; z) \\ F(1-i\lambda; -z) & \omega_- F(1-i\lambda; z) \end{pmatrix}.$$

If $f(t)$ *is a square integrable function of real* t *which vanishes outside of* $(-1, 1)$, *then the vector valued function* $F(z)$ *defined by*

$$\pi F_+(z) = \int_0^\infty t^{\frac{1}{2}i\lambda} F(1+i\lambda; -tz) f(t)\, dt + \omega_+ \int_{-\infty}^0 (-t)^{\frac{1}{2}i\lambda} F(1+i\lambda; -tz) f(t)\, dt$$

and

$$\pi F_-(z) = \int_0^\infty t^{-\frac{1}{2}i\lambda} F(1-i\lambda; -tz) f(t)\, dt + \omega_- \int_{-\infty}^0 (-t)^{-\frac{1}{2}i\lambda} F(1-i\lambda; -tz) f(t)\, dt$$

belongs to $\mathscr{H}(A, B)$, *and*

$$\pi \|F(t)\|^2 = \int_{-\infty}^{+\infty} |f(t)|^2\, dt.$$

Every element of $\mathscr{H}(A, B)$ *is of this form. Let* $f(t)$ *and* $g(t)$ *be square integrable functions of real* t *which vanish outside of* $(-1, 1)$, *and let* $F(z)$ *and* $G(z)$ *be the corresponding elements of* $\mathscr{H}(A, B)$. *The condition* $G(z) = zF(z)$ *is necessary and sufficient that* $f(t)$ *be (equivalent to) a differentiable function of* $t \neq 0$, *with absolutely continuous derivative, such that*

$$g(t) = tf''(t) + f'(t) + \tfrac{1}{4}\lambda^2 f(t)/t$$

for almost all t, *such that*

$$\lim_{t\nearrow 0}[tf'(t) - \tfrac{1}{2}i\lambda f(t)]\,(-t)^{\frac{1}{2}i\lambda} = \omega_+ \lim_{t\searrow 0}[tf'(t) - \tfrac{1}{2}i\lambda f(t)]\, t^{\frac{1}{2}i\lambda},$$

and such that

$$\lim_{t\nearrow 0}[tf'(t) + \tfrac{1}{2}i\lambda f(t)]\,(-t)^{-\frac{1}{2}i\lambda} = \omega_- \lim_{t\searrow 0}[tf'(t) + \tfrac{1}{2}i\lambda f(t)]\, t^{-\frac{1}{2}i\lambda}.$$

The condition $G(z) = a^{\frac{1}{2}+\frac{1}{2}\nu} F(az)$ *is necessary and sufficient that* $g(t) = a^{-\frac{1}{2}} f(t/a)$ *almost everywhere. The condition* $G(z) = zF''(z) + (1+\nu) F'(z)$ *is necessary and sufficient that* $g(t) = tf(t)$ *almost everywhere. The identity*

$$\begin{aligned} \pi \| F(t)\|^2/[2\cosh(\pi\lambda) - \omega_+\bar\omega_- - \omega_-\bar\omega_+] \\ = \int_{-\infty}^{0} | F_+(t)/\Gamma(1+i\lambda) - F_-(t)/\Gamma(1-i\lambda)|^2\,dt \\ + \int_0^\infty | \bar\omega_+ F_+(t)/\Gamma(1+i\lambda) - \bar\omega_- F_-(t)/\Gamma(1-i\lambda)|^2\,dt, \end{aligned}$$

holds for every element $F(z)$ *of the space.*

A mean square form of the expansion follows.

THEOREM 6. *Let* λ *be a nonzero real number, let* ω_+ *and* ω_- *be distinct numbers of absolute value one, and assume a choice of square root made for* $\omega_+\omega_-$. *For each square integrable function* $f(x)$ *of real* x, *there exists a corresponding square integrable function* $g(x)$ *of real* x,

$$\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \int_{-\infty}^{+\infty} |g(t)|^2\,dt,$$

such that

$$\begin{aligned} &[2\cosh(\pi\lambda) - \omega_+\bar\omega_- - \omega_-\bar\omega_+]^{\frac{1}{2}} g(x) \\ &= i(\omega_-\bar\omega_+)^{\frac{1}{2}}/\Gamma(1+i\lambda) \int_0^\infty (xt)^{\frac{1}{2}i\lambda} F(1+i\lambda; -xt) f(t)\,dt \\ &\quad - i(\omega_+\bar\omega_-)^{\frac{1}{2}}/\Gamma(1-i\lambda) \int_0^\infty (xt)^{-\frac{1}{2}i\lambda} F(1-i\lambda; -xt) f(t)\,dt \\ &\quad + i(\omega_+\omega_-)^{\frac{1}{2}}/\Gamma(1+i\lambda) \int_{-\infty}^0 (-xt)^{\frac{1}{2}i\lambda} F(1+i\lambda; -xt) f(t)\,dt \\ &\quad - i(\omega_+\omega_-)^{\frac{1}{2}}/\Gamma(1-i\lambda) \int_{-\infty}^0 (-xt)^{-\frac{1}{2}i\lambda} F(1-i\lambda; -xt) f(t)\,dt \end{aligned}$$

for almost all $x > 0$, *such that*

$$\begin{aligned} &[2\cosh(\pi\lambda) - \omega_+\bar\omega_- - \omega_-\bar\omega_+]^{\frac{1}{2}} g(x) \\ &= i(\bar\omega_+\bar\omega_-)^{\frac{1}{2}}/\Gamma(1+i\lambda) \int_0^\infty (-xt)^{\frac{1}{2}i\lambda} F(1+i\lambda; -xt) f(t)\,dt \\ &\quad - i(\bar\omega_+\bar\omega_-)^{\frac{1}{2}}/\Gamma(1-i\lambda) \int_0^\infty (-xt)^{-\frac{1}{2}i\lambda} F(1-i\lambda; -xt) f(t)\,dt \\ &\quad + i(\omega_+\bar\omega_-)^{\frac{1}{2}}/\Gamma(1+i\lambda) \int_{-\infty}^0 (xt)^{\frac{1}{2}i\lambda} F(1+i\lambda; -xt) f(t)\,dt \\ &\quad - i(\omega_-\bar\omega_+)^{\frac{1}{2}}/\Gamma(1-i\lambda) \int_{-\infty}^0 (xt)^{-\frac{1}{2}i\lambda} F(1-i\lambda; -xt) f(t)\,dt \end{aligned}$$

for almost all $x < 0$, and such that the same formulas hold with $f(x)$ and $g(x)$ interchanged, the integrals being mean square limits of $\int_{-a}^{a}$.

The Hankel transformation is related to the representations of the group of two-by-two matrices with real entries and determinant one [3]. Let a, b, and c be numbers, neither a nor b a nonpositive integer, such that $a + \bar{b} = c$ and $c \geqslant 1$. Then there exists a unique Hilbert space $\mathscr{F}(a, b; c; z)$, whose elements are functions defined in the upper half-plane, such that the expression

$$K(w, z) = \frac{(i\bar{w} - iw)^{a+b-c} \, |\Gamma(a) \, \Gamma(b)|^2}{(i\bar{w} - iz)^a \, (i\bar{z} - iw)^b} F\left(a, b; c; \frac{z - w}{z - \bar{w}} \frac{\bar{z} - \bar{w}}{\bar{z} - w}\right)$$

belongs to the space as a function of z when w is in the upper half-plane and such that the identity $F(w) = \langle F(z), K(w, z)\rangle$ holds for every element $F(z)$ of the space. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a matrix with real entries and determinant one and if a continuous choice of argument is made for $Cz + D$, then the transformation

$$F(z) \to \frac{1}{(Cz + D)^a \, (C\bar{z} + D)^b} F\left(\frac{Az + B}{Cz + D}\right)$$

is an isometry of the space onto itself. The space is now used only in the case $c = 1$. If $f(x)$ is a square integrable function of real x, there exists a corresponding element $F(z)$ of $\mathscr{F}(a, b; 1; z)$ such that the identity

$$\begin{aligned} 2\pi F(w) = {} & \Gamma(1 - a) \, \Gamma(1 - b) \, (i\bar{w} - iw)^{1-a-b} \\ & \times \int_{-\infty}^{+\infty} f(t) \, (i\bar{w} - it)^{a-1} \, (it - iw)^{b-1} \, dt \end{aligned}$$

holds when w is in the upper half-plane, and such that

$$2\pi \, \| F(z) \|^2 = \int_{-\infty}^{+\infty} | f(t) |^2 \, dt.$$

Every element of $\mathscr{F}(a, b; 1; z)$ is of this form. The identity

$$0 = (z - \bar{z})^2 \, \partial^2 F/\partial z \, \partial \bar{z} - a(\bar{z} - z) \, \partial F/\partial \bar{z} - b(z - \bar{z}) \, \partial F/\partial z$$

holds for every element $F(z)$ of the space. Define transformations L_+, L_-, and D in the space by $D : F(z) \to G(z)$ if $F(z)$ and $G(z)$ are elements of the space such that

$$\begin{aligned} G(z) = {} & \tfrac{1}{2} \, i(z^2 + 1) \, \partial F/\partial z + \tfrac{1}{2} \, iazF(z) \\ & + \tfrac{1}{2} \, i(\bar{z}^2 + 1) \, \partial F/\partial \bar{z} + \tfrac{1}{2} \, ib\bar{z}F(z), \end{aligned}$$

by $L_- : F(z) \to G(z)$ if $F(z)$ and $G(z)$ are elements of the space such that

$$\begin{aligned} G(z) = {} & \tfrac{1}{2} \, i(z - i)^2 \, \partial F/\partial z + \tfrac{1}{2} \, ia(z - i) F(z) \\ & + \tfrac{1}{2} \, i(\bar{z} - i)^2 \, \partial F/\partial \bar{z} + \tfrac{1}{2} \, ib(\bar{z} - i) F(z), \end{aligned}$$

and by $L_+ : F(z) \to G(z)$ if $F(z)$ and $G(z)$ are elements of the space such that

$$G(z) = \tfrac{1}{2} i(z+i)^2 \, \partial F/\partial z + \tfrac{1}{2} ia(z+i) F(z) + \tfrac{1}{2} i(\bar{z}+i)^2 \, \partial F/\partial \bar{z} + \tfrac{1}{2} ib(\bar{z}+i) F(z).$$

Then L_+ and L_- are adjoints, and D is self-adjoint. The identities

$$\begin{aligned} DL_- - L_-D &= -L_-, \\ DL_+ - L_+D &= L_+, \\ L_-L_+ - L_+L_- &= 2D \end{aligned}$$

are satisfied formally. The space admits an orthonormal basis consisting of the functions

$$\frac{a(a+1)\cdots(a+n-1)}{n!} \frac{(1+iz)^n}{(1-iz)^{a+n}(1+i\bar{z})^b} \times F\left(a+n, b; n+1; \frac{z-i}{z+i}\frac{\bar{z}+i}{\bar{z}-i}\right)$$

and

$$\frac{b(b+1)\cdots(b+n-1)}{n!} \frac{(1+iz)^n}{(1-iz)^{a}(1+i\bar{z})^{b+n}} \times F\left(a, b+n; n+1; \frac{z-i}{z+i}\frac{\bar{z}+i}{\bar{z}-i}\right)$$

for positive integral n and the function

$$\frac{1}{(1-iz)^a(1+i\bar{z})^b} F\left(a, b; 1; \frac{z-i}{z+i}\frac{\bar{z}+i}{\bar{z}-i}\right).$$

The square of the norm of each function is $\frac{1}{2} \mid \Gamma(a)\,\Gamma(b) \mid^{-2}$. An expansion holds for the elements of the space.

THEOREM 7. *Let a and b be nonintegral numbers such that $a + \bar{b} = 1$. If $f(x)$ is a square integrable function of real x, there exists a corresponding element $F(z)$ of $\mathscr{F}(a, b; 1; z)$ such that*

$$\begin{aligned} F(z) = {} & \int_0^\infty f(t)\, t^{\frac{1}{2}(a+b-1)} \exp(\tfrac{1}{2}itz + \tfrac{1}{2}it\bar{z}) \\ & \times (it\bar{z} - itz)^{-\frac{1}{2}(a+b)}\, \Gamma(b)\, W_{\frac{1}{2}(a-b), \frac{1}{2}(a+b-1)}(it\bar{z} - itz)\, dt \\ & + \int_{-\infty}^0 f(t)\,(-t)^{\frac{1}{2}(a+b-1)} \exp(\tfrac{1}{2}itz + \tfrac{1}{2}it\bar{z}) \\ & \times (itz - it\bar{z})^{-\frac{1}{2}(a-b)}\, \Gamma(a)\, W_{\frac{1}{2}(b-a), \frac{1}{2}(a+b-1)}(itz - it\bar{z})\, dt \end{aligned}$$

and such that

$$\|F(z)\|^2 = \int_{-\infty}^{+\infty} |f(t)|^2\, dt.$$

Every element of $\mathscr{F}(a, b; 1; z)$ is of this form. Let $f(x)$ and $g(x)$ be square integrable functions of real x, and let $F(z)$ and $G(z)$ be the corresponding elements of $\mathscr{F}(a, b; 1; z)$. The condition

$$G(z) = \partial F/\partial z + \partial F/\partial \bar{z}$$

holds if, and only if, $g(x) = ixf(x)$ almost everywhere. The condition

$$G(z) = z\partial F/\partial z + \tfrac{1}{2}\, aF(z) + \bar{z}\partial F/\partial \bar{z} + \tfrac{1}{2}\, bF(z)$$

holds if, and only if, $f(x)$ is equivalent to an absolutely continuous function of $x \neq 0$ such that

$$g(x) = -\, xf'(x) - \tfrac{1}{2} f(x)$$

almost everywhere. The condition

$$G(z) = z^2\, \partial F/\partial z + azF(z) + \bar{z}^2\, \partial F/\partial \bar{z} + b\bar{z}F(z)$$

holds if, and only if, $f(x)$ is equivalent to a differentiable function of $x \neq 0$ with absolutely continuous derivative such that

$$g(x) = -\, ixf''(x) - if'(x) + \tfrac{1}{4}i(1 - a - b)^2 f(x)/x$$

almost everywhere, such that

$$\begin{aligned}&\lim_{x \nearrow 0}[xf'(x) - \tfrac{1}{2}i\lambda f(x)]\,(-\,x)^{\frac{1}{2}i\lambda}\\ &\qquad = \sin(\pi a)/\sin(\pi b) \lim_{x \searrow 0}[xf'(x) - \tfrac{1}{2}i\lambda f(x)]\; x^{\frac{1}{2}i\lambda},\end{aligned}$$

and such that

$$\lim_{x \nearrow 0}[xf'(x) + \tfrac{1}{2}i\lambda f(x)]\,(-\,x)^{-\frac{1}{2}i\lambda} = \lim_{x \searrow 0}[xf'(x) + \tfrac{1}{2}i\lambda f(x)]\; x^{-\frac{1}{2}i\lambda}.$$

Relevant are some properties of the hypergeometric function with argument one-half.

THEOREM 8. *The identities*

$$\begin{aligned}&4c(c - a - b)F(a, b; c; \tfrac{1}{2})F(-\, a, -\, b; c - a - b; \tfrac{1}{2})\\ &\qquad + abF(a + 1, b + 1; c + 1; \tfrac{1}{2})F(1 - a, 1 - b; c + 1 - a - b; \tfrac{1}{2})\\ &\quad = 4c(c - a - b)\, \Gamma(c)\, \Gamma(c - a - b)/[\Gamma(c - a)\, \Gamma(c - b)]\end{aligned}$$

and

$$4c(1-c)F(a,b;c;\tfrac12)F(-a,-b;1-c;\tfrac12) - abF(a+1,b+1;c+1;\tfrac12)F(1-a,1-b;2-c;\tfrac12) = 4c(1-c)\sin(\pi c-\pi a-\pi b)/\sin(\pi c)$$

and

$$cF(a,b;c;\tfrac12)F(a+1,b+1;a+b+2-c;\tfrac12) + (a+b+1-c)F(a+1,b+1;c+1;\tfrac12)F(a,b;a+b+1-c;\tfrac12) = 2^{a+b+1}\Gamma(c+1)\,\Gamma(a+b+2-c)/[\Gamma(a+1)\,\Gamma(b+1)]$$

are satisfied when c and $a+b-c$ are not integers.

A computation of Hankel transforms is obtained in terms of Mellin transforms.

THEOREM 9. *Let a and b be nonreal numbers such that $a+\bar{b}=1$. Choose $\lambda = i(a+b-1)$, $\omega_- = 1$, and $\omega_+ = \sin(\pi a)/\sin(\pi b)$ with the square root of $\omega_+\omega_-$ to be $\sin(\pi a)/|\sin(\pi a)|$ when $\lambda < 0$ and $-\sin(\pi a)/|\sin(\pi a)|$ when $\lambda > 0$. Define functions $P(s)$, $Q(s)$, $R(s)$, and $S(s)$ by*

$$\Gamma(s+\tfrac12-\tfrac12a+\tfrac12b)\,P(s) = \Gamma(s-\tfrac12 i\lambda)\,\Gamma(s+\tfrac12 i\lambda)\,2^{\frac12a-\frac12b}\Gamma(b) \times F(1-a,b;s+\tfrac12-\tfrac12a+\tfrac12b;\tfrac12),$$

$$\Gamma(s+\tfrac12+\tfrac12a-\tfrac12b)\,Q(s) = \Gamma(s-\tfrac12 i\lambda)\,\Gamma(s+\tfrac12 i\lambda)\,2^{-\frac12a+\frac12b}\Gamma(a) \times F(a,1-b;s+\tfrac12+\tfrac12a-\tfrac12b;\tfrac12),$$

$$\Gamma(s+\tfrac32-\tfrac12a+\tfrac12b)\,R(s) = \Gamma(s-\tfrac12 i\lambda)\,\Gamma(s+\tfrac12 i\lambda)\,2^{\frac12a-\frac12b-1}\Gamma(b+1) \times F(2-a,b+1;s+\tfrac32-\tfrac12a+\tfrac12b;\tfrac12),$$

$$\Gamma(s-\tfrac12+\tfrac12a-\tfrac12b)\,S(s) = \Gamma(s-\tfrac12 i\lambda)\,\Gamma(s+\tfrac12 i\lambda)\,2^{1-\frac12a+\frac12b}\Gamma(a-1) \times F(a-1,-b;s-\tfrac12+\tfrac12a-\tfrac12b;\tfrac12)$$

when $\operatorname{Re} s > 0$. *A necessary and sufficient condition that square integrable functions $f(x)$ and $g(x)$ of real x be related as in Theorem 6 is that the mean square Mellin transforms be related by the equations*

$$P(1-s)\int_0^\infty g(t)\,t^{s-1}\,dt + Q(1-s)\int_0^\infty g(-t)\,t^{s-1}\,dt = P(s)\int_0^\infty f(t)\,t^{-s}\,dt + Q(s)\int^\infty f(-t)\,t^{-s}\,dt,$$

and

$$R(1-s)\int_0^\infty g(t)\,t^{s-1}\,dt + S(1-s)\int_0^\infty g(-t)\,t^{s-1}\,dt$$
$$= -R(s)\int_0^\infty f(t)\,t^{-s}\,dt - S(s)\int_0^\infty f(-t)\,t^{-s}\,dt$$

almost everywhere on the line $\operatorname{Re} s = \frac{1}{2}$.

Another computation of Hankel transforms follows.

THEOREM 10. *Let a and b be nonreal numbers such that $a + \bar{b} = 1$. Choose $\lambda = i(a + b - 1)$, $\omega_- = 1$, and $\omega_+ = \sin(\pi a)/\sin(\pi b)$ with the square root of $\omega_+\omega_-$ to be $\sin(\pi a)/|\sin(\pi a)|$ if $\lambda < 0$ and $-\sin(\pi a)/|\sin(\pi a)|$ is $\lambda > 0$. Let $f(x)$ and $g(x)$ be square integrable functions of real x, and let $F(z)$ and $G(z)$ be the corresponding elements of $\mathcal{F}(a, b; 1; z)$ defined as in Theorem 7. A necessary and sufficient condition that $f(x)$ and $g(x)$ be related as in Theorem 6 is that*

$$G(z) = (-iz)^{-a}\,(i\bar{z})^{-b}\,F(-1/z)$$

in the upper half-plane.

Proof of Theorem 1. The proof follows the proof of Theorem 50 of [2]. The hypotheses imply that the transformation $F(z) \to F(w)$ takes $\mathcal{H}(A, B)$ onto $\mathcal{C}$ for every complex number w. For every positive number a, there exists a space $\mathcal{H}(A(a), B(a))$ such that the transformation $F(z) \to a^{\frac{1}{2}+\frac{1}{2}\nu}F(az)$ is an isometry of $\mathcal{H}(A, B)$ onto $\mathcal{H}(A(a), B(a))$. The space $\mathcal{H}(A(a), B(a))$ is contained isometrically in the space $\mathcal{H}(A(b), B(b))$ when $a < b$, and the identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\,M(a, b, z)$$

holds for a unique matrix $M(a, b, z)$ of operator valued entire functions such that a space $\mathcal{H}(M(a, b))$ exists. The choice of $A(a, z)$ and $B(a, z)$ can be made in a unique way such that $A(a, z) = A(z)$ and $B(a, z) = B(z)$ when $a = 1$ and so that the value of $M(a, b, z)$ at the origin is always the identity matrix. The identity

$$(A(a, z), B(a, z)) = a^{\frac{1}{2}\nu}(A(a, z), B(a, z))\,P(a)$$

holds for a unique matrix

$$P(a) = \begin{pmatrix} p(a) & q(a) \\ r(a) & s(a) \end{pmatrix}$$

of operators such that $P(a)\,I\bar{P}(a) = I$. The identity

$$b^{\frac{1}{2}\nu}(A(b, z), B(b, z))\,P(b) = a^{\frac{1}{2}\nu}(A(a, z), B(a, z))\,P(a)\,M(a, b, z),$$

which holds when $a < b$, implies that

$$(A(z), B(z))\, P(b) = (a/b)^{\frac{1}{2}\nu}\, (A(az/b), B(az/b))\, P(a)\, M(a, b, z/b)$$

and that

$$P(a/b)\, M(a/b, 1, z)\, P(b) = P(a)\, M(a, b, z/b).$$

When $z = 0$, the identity reads $P(a/b)\, P(b) = P(a)$. There exists a non-decreasing function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \bar{\beta}(t) & \gamma(t) \end{pmatrix}$$

of $t > 0$, whose values are self-adjoint matrices of operators, such that the identity

$$M'(a, b, 0)\, I = m(b) - m(a)$$

holds when $a < b$. The integral equation

$$M(a, b, w)\, I - I = w \int_a^b M(a, t, w)\, dm(t)$$

holds when $a < b$. The identity

$$b[m(1) - m(a/b)] = P(b)\, [m(b) - m(a)]\, \bar{P}(b)$$

is obtained by taking derivatives at the origin in the identity for $M(a, b, z)$. It follows that $m(t)$ is a differentiable function of t and that

$$P(t)\, m'(t)\, \bar{P}(t) = m'(1).$$

The integral equation

$$(A(b, z), B(b, z))\, I - (A(a, z), B(a, z))\, I$$
$$= z \int_a^b (A(t, z), B(t, z))\, dm(t),$$

which holds when $a < b$, implies that

$$zB'(z) + \tfrac{1}{2}\nu B(z) + A(z)\, q'(1) + B(z)\, s'(1)$$
$$= zA(z)\, \alpha'(1) + zB(z)\, \beta'(1)$$

and that

$$-zA'(z) - \tfrac{1}{2}\nu A(z) - A(z)\, p'(1) - B(z)\, r'(1)$$
$$= zA(z)\, \beta'(1) + zB(z)\, \gamma'(1).$$

Since $A(z)$ and $B(z)$ are chosen so that $A(0) = 1$ and $B(0) = 0$, it follows that $p(t) = t^{-\frac{1}{2}\nu}$, $q(t) = 0$, and $s(t) = t^{\frac{1}{2}\bar{\nu}}$. The condition $P(t)\, I\bar{P}(t) = I$ implies that $r(t)\, t^{\frac{1}{2}\nu}$ is self-adjoint. Since $r(1) = 0$, it follows that $r'(1)$ is self-adjoint. Since the transformation $F(z) \to F(0)$ takes $\mathscr{H}(A, B)$ onto $\mathscr{C}$, the operator $B'(0)$ has no nonzero vector in its kernel. It follows that $\alpha'(0)$ has no nonzero vector in its kernel.

Proof of Theorem 2. When $F(z) = K(\alpha, z)\, a$ and $G(z) = K(\beta, z)\, b$ for numbers α and β and vectors a and b, the second identity in the statement of the theorem reads

$$\beta\bar{b}F''(\beta) + \bar{b}(1 + \nu) F'(\beta) = \bar{\alpha}\bar{G}''(\alpha)\, a + \bar{G}'(\alpha)\,(1 + \bar{\nu})\, a.$$

An equivalent identity is

$$\begin{aligned}
&[z^2B''(z) + (\nu - 1)\, zB'(z) - \nu B(z)]\, \bar{A}(w) \\
&\quad - [z^2A''(z) + (\nu - 1)\, zA'(z) - \nu A(z)]\, \bar{B}(w) \\
&\quad - [zB''(z) + (1 + \nu)\, B'(z)]\, \bar{w}\bar{A}(w) \\
&\quad + [zA''(z) + (1 + \nu)\, A'(z)]\, \bar{w}\bar{B}(w) \\
&= A(z)\, [\bar{w}^2\bar{B}''(w) + \bar{w}\bar{B}'(w)\,(\bar{\nu} - 1) - \bar{B}(w)\, \bar{\nu}] \\
&\quad - B(z)\, [\bar{w}^2\bar{A}''(w) + \bar{w}\bar{A}'(w)\,(\bar{\nu} - 1) - \bar{A}(w)\, \bar{\nu}] \\
&\quad - zA(z)\, [\bar{w}\bar{B}''(w) + \bar{B}'(w)\,(1 + \bar{\nu})] \\
&\quad + zB(z)\, [\bar{w}\bar{A}''(w) + \bar{A}'(w)\,(1 + \bar{\nu})].
\end{aligned}$$

By the computation of derivatives in the statement of Theorem 1, the identity reads

$$\begin{aligned}
&A(z)\, [\alpha'(1)\, \bar{\beta}'(1) - \beta'(1)\, \alpha'(1)]\, \bar{A}(w) \\
&\quad + A(z)\, [\alpha'(1)\, \gamma'(1) - \beta'(1)^2]\, \bar{B}(w) \\
&\quad + B(z)\, [\bar{\beta}'(1)^2 - \gamma'(1)\, \alpha'(1)]\, \bar{A}(w) \\
&\quad + B(z)\, [\bar{\beta}'(1)\, \gamma'(1) - \gamma'(1)\, \beta'(1)]\, \bar{B}(w) = 0.
\end{aligned}$$

By the arbitrariness of $A(z)$ and $B(z)$, it follows that

$$\begin{aligned}
\beta'(1)\, \alpha'(1) &= \alpha'(1)\, \bar{\beta}'(1), \\
\gamma'(1)\, \beta'(1) &= \bar{\beta}'(1)\, \gamma'(1), \\
\alpha'(1)\, \gamma'(1) &= \beta'(1)^2.
\end{aligned}$$

These equations imply that

$$\alpha'(1) = U\bar{U}, \qquad \beta'(1) = U\bar{V}, \qquad \gamma'(1) = V\bar{V}$$

for operators U and V such that $\bar{U}V = \bar{V}U$. A similar treatment of the first identity in the statement of the theorem will show that $K(w, z)$ commutes with ν^2. It follows that $A(z)$, $B(z)$, $\alpha'(1)$, $\beta'(1)$, and $\gamma'(1)$ commute with ν^2. So U and V can be chosen to commute with ν^2. A straightforward calculation will show that the equation

$$z\Phi''(z) + (1 + \nu)\,\Phi'(z) + \Phi(z)\,(\bar{V}\nu U + \bar{U}\bar{\nu}V)/2 = 0$$

is satisfied. The stated power series expansion follows.

Proof of Theorem 3. Define $\Phi(z)$ to be the sum of the power series in the statement of Theorem 2. Define $B(z)$ by the integral representation

$$B(z) = z \int_0^1 t^{\frac{1}{2}\nu}\Phi(tz)\,\bar{U}t^{\frac{1}{2}\bar{\nu}}\,dt.$$

Define $A(z)$ so that the identity

$$\Phi(z) = A(z)\,U + B(z)\,V$$

is satisfied. These conditions imply that

$$zB'(z) + \tfrac{1}{2}\,\nu B(z) + \tfrac{1}{2}\,B(z)\,\bar{\nu} = z[A(z)\,U + B(z)\,V]\,\bar{V}$$

and that

$$\begin{aligned} &z[A(z)\,U + B(z)\,V]'' + (1 + \nu)\,[A(z)\,U + B(z)\,V]' \\ &\qquad + [A(z)\,U + B(z)\,V]\,(\bar{V}\nu U + \bar{U}\bar{\nu}V)/2 = 0. \end{aligned}$$

If $C(z)$ is defined by the identity

$$-\,zA'(z) - \tfrac{1}{2}\,\nu A(z) + \tfrac{1}{2}\,A(z)\,\nu - C(z) = z[A(z)\,U + B(z)\,V]\,\bar{V},$$

then

$$zC'(z) + \tfrac{1}{2}\,\nu C(z) + \tfrac{1}{2}\,C(z)\,\nu = 0.$$

Since the leading coefficient in the power series expansion of $C(z)$ is zero, it follows that $C(z)$ vanishes identically. Choose $m(t)$ to be a solution of the equation

$$m'(t) = \begin{pmatrix} t^{\frac{1}{2}\nu} & 0 \\ 0 & t^{-\frac{1}{2}\bar{\nu}} \end{pmatrix} \begin{pmatrix} U\bar{U} & U\bar{V} \\ V\bar{U} & V\bar{V} \end{pmatrix} \begin{pmatrix} t^{\frac{1}{2}\bar{\nu}} & 0 \\ 0 & t^{-\frac{1}{2}\nu} \end{pmatrix}$$

with self-adjoint values for $t > 0$. If $A(a, z) = a^{\frac{1}{2}\nu}A(a, z)\,a^{-\frac{1}{2}\nu}$ and $B(a, z) = a^{\frac{1}{2}\nu}B(a, z)\,a^{\frac{1}{2}\bar{\nu}}$, a straightforward calculation will show that the integral equation

$$\begin{aligned} &B(a, z)\,\bar{A}(a, w) - A(a, z)\,\bar{B}(a, w) \\ &\qquad = (z - \bar{w}) \int_0^a (A(t, z), B(t, z))\,dm(t)\,(A(t, w), B(t, w))^- \end{aligned}$$

holds when $a > 0$. The existence of the space $\mathscr{H}(A, B)$ follows. The desired properties of the space are obtained by reversing steps in the proofs of Theorems 1 and 2.

Proof of Theorem 4. These results are obtained in a straightforward way from the general expansion theorem for Hilbert spaces of entire functions [1].

Proof of Theorem 5. The expansion is obtained by a straightforward calculation from the results of Theorem 4. The choice of U and V has been made in such a way that the equations

$$2J = i\lambda(\bar{V}JU - \bar{U}JV)$$

and $\bar{V}U = \bar{U}V$ are satisfied, where $\nu = i\lambda J$. The equations can be written

$$2(UJ\bar{U}) = i\lambda(UJ\bar{V})\,J(UJ\bar{U}) - i\lambda(UJ\bar{U})\,J(VJ\bar{U})$$

and

$$(UJ\bar{V})\,(UJ\bar{U}) = (UJ\bar{U})\,(VJ\bar{U}).$$

A solution is obtained with $i\lambda UJ\bar{V} = J$ if $UJ\bar{U}$ anticommutes with J. The formula for norms at the end of the theorem is established using Mellin transforms. Details are given in the proof of the next theorem.

Proof of Theorem 6. The proof depends on the identity

$$\int_0^\infty t^{s+\frac{1}{2}i\lambda-1}F(1 + i\lambda; -t)/\Gamma(1 + i\lambda)\,dt = \Gamma(s + \tfrac{1}{2}i\lambda)/\Gamma(1 + i\lambda - s),$$

which holds when $0 < \operatorname{Re} s < \frac{3}{4}$, and the identity

$$\int_0^\infty t^{s-1}[F(1 - i\lambda; t)/\Gamma(1 - i\lambda) - F(1 + i\lambda; t)/\Gamma(1 + i\lambda)]\,dt$$
$$= \Gamma(s - \tfrac{1}{2}i\lambda)\,\Gamma(s + \tfrac{1}{2}i\lambda)\sin(\pi i\lambda)/\pi$$

which holds when $\operatorname{Re} s > 0$. The theorem follows from the mean square theory of Mellin transforms since square integrable functions $f(x)$ and $g(x)$ of real x are related as in the statement of the theorem if, and only if, the identities

$$\int_0^\infty g(t)\,t^{s-1}\,dt = P(s)\int_0^\infty f(t)\,t^{-s}\,dt + Q(s)\int_0^\infty f(-t)\,t^{-s}\,dt$$

and

$$\int_0^\infty g(-t)\, t^{s-1}\, dt = R(s) \int_0^\infty f(t)\, t^{-s}\, dt + S(s) \int_0^\infty f(-t)\, t^{-s}\, dt$$

hold almost everywhere on the line Re $s = \frac{1}{2}$, where

$$\pi[2\cosh(\pi\lambda) - \omega_+\bar{\omega}_- - \omega_-\bar{\omega}_+]^{\frac{1}{2}} P(s)$$
$$= \Gamma(s - \tfrac{1}{2}i\lambda)\, \Gamma(s + \tfrac{1}{2}i\lambda)\, [i(\omega_-\bar{\omega}_+)^{\frac{1}{2}} \sin(\pi s - \tfrac{1}{2}\pi i\lambda) - i(\omega_+\bar{\omega}_-)^{\frac{1}{2}} \sin(\pi s + \tfrac{1}{2}\pi i\lambda)]$$

and

$$\pi[2\cosh(\pi\lambda) - \omega_+\bar{\omega}_- - \omega_-\bar{\omega}_+]^{\frac{1}{2}} Q(s)$$
$$= \Gamma(s - \tfrac{1}{2}i\lambda)\, \Gamma(s + \tfrac{1}{2}i\lambda)\, i(\omega_+\omega_-)^{\frac{1}{2}} \sin(\pi i\lambda)$$

and

$$\pi[2\cosh(\pi\lambda) - \omega_+\bar{\omega}_- - \omega_-\bar{\omega}_+]^{\frac{1}{2}} R(s)$$
$$= \Gamma(s - \tfrac{1}{2}i\lambda)\, \Gamma(s + \tfrac{1}{2}i\lambda)\, i(\bar{\omega}_+\bar{\omega}_-)^{\frac{1}{2}} \sin(\pi i\lambda)$$

and

$$\pi[2\cosh(\pi\lambda) - \omega_+\bar{\omega}_- - \omega_-\bar{\omega}_+]^{\frac{1}{2}} S(s)$$
$$= \Gamma(s - \tfrac{1}{2}i\lambda)\, \Gamma(s + \tfrac{1}{2}i\lambda)\, [i(\omega_+\bar{\omega}_-)^{\frac{1}{2}} \sin(\pi s - \tfrac{1}{2}\pi i\lambda) - i(\omega_-\bar{\omega}_+)^{\frac{1}{2}} \sin(\pi s + \tfrac{1}{2}\pi i\lambda)].$$

Proof of Theorem 7. By properties of orthogonal sets, the identity

$$\tfrac{1}{2}K(w, z)/h$$
$$= \sum_{n=0}^\infty \frac{\Gamma(a+n)\,\Gamma(b)\,(h+iz)^n}{\Gamma(n+1)\,(h-iz)^{a+n}\,(h+i\bar{z})^b} F\left(a+n, b; n+1; \frac{z-ih}{z+ih}\,\frac{\bar{z}+ih}{\bar{z}-ih}\right)$$
$$\times \frac{\Gamma(n+1-b)\,\Gamma(1-a)\,(h-i\bar{w})^n}{\Gamma(n+1)\,(h+i\bar{w})^{n+1-b}\,(h-iw)^{1-a}}$$
$$\times F\left(n+1-b, 1-a; n+1; \frac{w-ih}{w+ih}\,\frac{\bar{w}+ih}{\bar{w}-ih}\right)$$
$$+ \sum_{n=1}^\infty \frac{\Gamma(a)\,\Gamma(n+b)\,(h-i\bar{z})^n}{\Gamma(n+1)\,(h-iz)^a\,(h+i\bar{z})^{b+n}} F\left(a, b+n; n+1; \frac{z-ih}{z+ih}\,\frac{\bar{z}+ih}{\bar{z}-ih}\right)$$
$$\times \frac{\Gamma(1-b)\,\Gamma(n+1-a)\,(h+iw)^n}{\Gamma(n+1)\,(h+i\bar{w})^{1-b}\,(h-iw)^{n+1-a}}$$
$$\times F\left(1-b, n+1-a; n+1; \frac{w-ih}{w+ih}\,\frac{\bar{w}+ih}{\bar{w}-ih}\right)$$

holds in the space $\mathscr{F}(a, b; 1; z)$ when $h = 1$. The identity follows by a change of variable for any positive h. By Euler's representation of the hypergeometric function, the identity can be written

$$\begin{aligned}
\tfrac{1}{2}hK(w, z) = {} & \frac{(2i\bar{z} - 2iz)^{-b}(2i\bar{w} - 2iw)^{a-1}}{(1 - iz/h)^{a-b}(1 + i\bar{w}/h)^{a-b}} \sum_{n=0}^{\infty} \frac{(1 + iz/h)^n (1 - i\bar{w}/h)^n}{(1 - iz/h)^n (1 + i\bar{w}/h)^n} \\
& \times \int_0^h t^{-a}(1 - t/h)^{a+n-1} \left\{1 + \frac{t(1 + iz/h)(1 - i\bar{z}/h)}{2i\bar{z} - 2iz}\right\}^{-b} dt \\
& \times \int_0^h t^{b-1}(1 - t/h)^{n-b} \left\{1 + \frac{t(1 + iw/h)(1 - i\bar{w}/h)}{2i\bar{w} - 2iw}\right\}^{a-1} dt \\
& + \frac{(2i\bar{z} - 2iz)^{-a}(2i\bar{w} - 2iw)^{b-1}}{(1 + i\bar{z}/h)^{b-a}(1 - iw/h)^{b-a}} \sum_{n=1}^{\infty} \frac{(1 - i\bar{z}/h)^n (1 + iw/h)^n}{(1 + i\bar{z}/h)^n (1 - iw/h)^n} \\
& \times \int_0^h t^{-b}(1 - t/h)^{b+n-1} \left\{1 + \frac{t(1 + iz/h)(1 - i\bar{z}/h)}{2i\bar{z} - 2iz}\right\}^{-a} dt \\
& \times \int_0^h t^{a-1}(1 - t/h)^{n-a} \left\{1 + \frac{t(1 + iz/h)(1 - i\bar{z}/h)}{2i\bar{z} - 2iz}\right\}^{b-1} dt.
\end{aligned}$$

In the limit of small h, the identity reads

$$\begin{aligned}
\tfrac{1}{2}K(w, z) = {} & \int_0^\infty \exp(itz + it\bar{z})(2it\bar{z} - 2itz)^{-\frac{1}{2}(a-b)} \\
& \times \Gamma(b)\, W_{\frac{1}{2}(a-b),\frac{1}{2}(a+b-1)}(2it\bar{z} - 2itz) \\
& \times \exp(-itw - it\bar{w})(2it\bar{w} - 2itw)^{\frac{1}{2}(a+b-2)} \\
& \times \Gamma(1 - a)\, W_{\frac{1}{2}(a-b),\frac{1}{2}(a+b-1)}(2it\bar{w} - 2itw)\, dt \\
& + \int_0^\infty \exp(-itz - it\bar{z})(2it\bar{z} - 2itz)^{-\frac{1}{2}(a-b)} \\
& \times \Gamma(a)\, W_{\frac{1}{2}(b-a),\frac{1}{2}(a+b-1)}(2it\bar{z} - 2itz) \\
& \times \exp(itw + it\bar{w})(2it\bar{w} - 2itw)^{\frac{1}{2}(a+b-2)} \\
& \times \Gamma(1 - b)\, W_{\frac{1}{2}(b-a),\frac{1}{2}(a+b-1)}(2it\bar{w} - 2itw)\, dt.
\end{aligned}$$

The theorem follows by straightforward arguments.

Proof of Theorem 8. The proof depends on the identities

$$\begin{aligned}
& (c - a)(c - b)\,\Gamma(c + 1)^{-1} F(a, b; c + 1; z) \\
& \quad = (c - a - b)\,\Gamma(c)^{-1} F(a, b; c; z) \\
& \qquad + ab(1 - z)\,\Gamma(c + 1)^{-1} F(a + 1, b + 1; c + 1; z)
\end{aligned}$$

and

$$(c-a)(c-b)z\Gamma(c+2)^{-1}F(a+1,b+1;c+2;z)$$
$$= \Gamma(c)^{-1}F(a,b;c;z) - c(1-z)\Gamma(c+1)^{-1}F(a+1,b+1;c+1;z)$$

for the hypergeometric series, and the identity

$$\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})F(a,b;c;\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}c),$$

which holds when $a+b+1=2c$. For any fixed a and b, the expression

$$\begin{aligned}&4\Gamma(c-a)\,\Gamma(c-b)\,\Gamma(c)^{-1}F(a,b;c;\tfrac{1}{2})\,\Gamma(c-a-b)^{-1}\\ &\quad\times F(-a,-b;c-a-b;\tfrac{1}{2}) + ab\Gamma(c-a)\,\Gamma(c-b)\,\Gamma(c+1)^{-1}\\ &\quad\times F(a+1,b+1;c+1;\tfrac{1}{2})\,\Gamma(c+1-a-b)^{-1}\\ &\quad\times F(1-a,1-b;c+1-a-b;\tfrac{1}{2})\end{aligned}$$

is a periodic entire function of c of period one which has value 4 when $a+b+1=2c$. By Euler's representation of the hypergeometric series, the expression is of bounded type and of nonpositive mean type in the upper and lower half-planes, and so is a constant. The first identity in the statement of the theorem follows. For any fixed a and b, the expression

$$\begin{aligned}&\Gamma(c)^{-1}F(a,b;c;\tfrac{1}{2})\,\Gamma(a+b+2-c)^{-1}F(a+1,b+1;a+b+2-c;\tfrac{1}{2})\\ &\quad+\Gamma(c+1)^{-1}F(a+1,b+1;c+1;\tfrac{1}{2})\\ &\quad\times\Gamma(a+b+1-c)^{-1}F(a,b;a+b+1-c;\tfrac{1}{2})\end{aligned}$$

is a periodic entire function of c of period one which has the value $2^{a+b+1}\Gamma(a+1)^{-1}\,\Gamma(b+1)^{-1}$ when $a+b+1=2c$. Since the expression is of bounded type and of mean type at most π in the upper and lower half-planes, it is a constant. The third identity in the statement of the theorem follows. For any fixed a and b, the expression

$$\begin{aligned}&4\Gamma(c)^{-1}F(a,b;c;\tfrac{1}{2})\,\Gamma(1-c)^{-1}F(-a,-b;1-c;\tfrac{1}{2})\\ &\quad-ab\Gamma(c+1)^{-1}F(a+1,b+1;c+1;\tfrac{1}{2})\\ &\quad\times\Gamma(2-c)^{-1}F(1-a,1-b;2-c;\tfrac{1}{2})\end{aligned}$$

is an entire function of c which changes sign when c is replaced by $c+1$, and which has the same value as $4\sin(\pi c-\pi a+\pi b)/\pi$ when $a+b+1=2c$. Since the expression is of bounded type and of mean type at most π in the upper and lower half-planes, it differs from $4\sin(\pi c-\pi a+\pi b)/\pi$ by a constant multiple of $\cos(\pi c - \frac{1}{2}\pi a - \frac{1}{2}\pi b)$. The second identity in the statement of the theorem follows from the identity

$$\begin{aligned}
&[4c(1-c)F(a,b;c;\tfrac{1}{2})F(-a,-b;1-c;\tfrac{1}{2})\\
&\quad - abF(a+1,b+1;c+1;\tfrac{1}{2})F(1-a,1-b;2-c;\tfrac{1}{2})]\\
&\quad \times [4(a+b+1-c)(c-a-b)F(a,b;a+b+1-c;\tfrac{1}{2})\\
&\quad \times F(-a,-b;c-a-b;\tfrac{1}{2}) - abF(a+1,b+1;a+b+2-c;\tfrac{1}{2})\\
&\quad \times F(1-a,1-b;c+1-a-b;\tfrac{1}{2})]\\
&\quad + 4ab[(1-c)F(-a,-b;1-c;\tfrac{1}{2})F(1-a,1-b;c+1-a-b;\tfrac{1}{2})\\
&\quad + (c-a-b)F(-a,-b;c-a-b;\tfrac{1}{2})F(1-a,1-b;2-c;\tfrac{1}{2})]\\
&\quad \times [cF(a,b;c;\tfrac{1}{2})F(a+1,b+1;a+b+2-c;\tfrac{1}{2})\\
&\quad + (a+b+1-c)F(a,b;a+b+1-c;\tfrac{1}{2})F(a+1,b+1;c+1;\tfrac{1}{2})]\\
&= [4c(c-a-b)F(a,b;c;\tfrac{1}{2})F(-a,-b;c-a-b;\tfrac{1}{2})\\
&\quad + abF(a+1,b+1;c+1;\tfrac{1}{2})F(1-a,1-b;c+1-a-b;\tfrac{1}{2})]\\
&\quad \times [4(1-c)(a+b+1-c)F(a,b;a+b+1-c;\tfrac{1}{2})\\
&\quad \times F(-a,-b;1-c;\tfrac{1}{2}) + abF(a+1,b+1;a+b+2-c;\tfrac{1}{2})\\
&\quad \times F(1-a,1-b;2-c;\tfrac{1}{2})].
\end{aligned}$$

Proof of Theorem 9. The theorem follows from the proof of Theorem 6 and the identity

$$\begin{aligned}
&\begin{pmatrix} P(1-s) & Q(1-s) \\ R(1-s) & S(1-s) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} P(s) & Q(s) \\ R(s) & S(s) \end{pmatrix}\\
&\quad = \frac{\Gamma(s-\frac{1}{2}i\lambda)\,\Gamma(s+\frac{1}{2}i\lambda)}{\pi}\\
&\quad\quad \times \begin{pmatrix} -\cos(\pi s + \frac{1}{2}\pi a - \frac{1}{2}\pi b) & \sin(\pi a) \\ \sin(\pi b) & -\cos(\pi s - \frac{1}{2}\pi a + \frac{1}{2}\pi b) \end{pmatrix}
\end{aligned}$$

which is obtained from Theorem 8. Do not confuse the present use of $P(s)$, $Q(s)$, $R(s)$, and $S(s)$ with the notation in the proof of Theorem 6.

Proof of Theorem 10. The theorem follows from Theorem 9 using the identity

$$\begin{aligned}
&\Gamma(s+\tfrac{1}{2}-k)\int_0^\infty t^{s-1}W_{k,m}(2t)\,t^{-\frac{1}{2}}\,dt\\
&\quad = 2^k\Gamma(s-m)\,\Gamma(s+m)F(\tfrac{1}{2}-k-m,\tfrac{1}{2}-k+m;s+\tfrac{1}{2}-k;\tfrac{1}{2})
\end{aligned}$$

which holds when $\operatorname{Re} s > 0$ if m is imaginary. Note also Theorem 6 of [3].

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